

**Non-collision and collision properties of  
Dyson's model in infinite dimension and other stochastic dynamics  
whose equilibrium states are determinantal random point fields**

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Dedicated to Professor Tokuzo Shiga on his 60th birthday

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**Abstract**

Dyson's model on interacting Brownian particles is a stochastic dynamics consisting of an infinite amount of particles moving in  $\mathbb{R}$  with a logarithmic pair interaction potential. For this model we will prove that each pair of particles never collide.

The equilibrium state of this dynamics is a determinantal random point field with the sine kernel. We prove for stochastic dynamics given by Dirichlet forms with determinantal random point fields as equilibrium states the particles never collide if the kernel of determining random point fields are locally Lipschitz continuous, and give examples of collision when Hölder continuous.

In addition we construct infinite volume dynamics (a kind of infinite dimensional diffusions) whose equilibrium states are determinantal random point fields. The last result is partial in the sense that we simply construct a diffusion associated with the *maximal closable part* of *canonical* pre Dirichlet forms for given determinantal random point fields as equilibrium states. To prove the closability of canonical pre Dirichlet forms for given determinantal random point fields is still an open problem. We prove these dynamics are the strong resolvent limit of finite volume dynamics.

## 1 Introduction

Dyson's model on interacting Brownian particles in infinite dimension is an infinitely dimensional diffusion process  $\{(X_t^i)_{i \in \mathbb{N}}\}$  formally given by the following stochastic differential equation (SDE):

$$dX_t^i = dB_t^i + \sum_{j=1, j \neq i}^{\infty} \frac{1}{X_t^i - X_t^j} dt \quad (i = 1, 2, 3, \dots), \quad (1.1)$$

where  $\{B_t^i\}$  is an infinite amount of independent one dimensional Brownian motions. The corresponding unlabeled dynamics is

$$\mathbb{X}_t = \sum_{i=1}^{\infty} \delta_{X_t^i}. \quad (1.2)$$

Here  $\delta \cdot$  denote the point mass at  $\cdot$ . By definition  $\mathbb{X}_t$  is a  $\Theta$ -valued diffusion, where  $\Theta$  is the set consisting of configurations on  $\mathbb{R}$ ; that is,

$$\Theta = \{\theta = \sum_i \delta_{x_i}; x_i \in \mathbb{R}, \theta(\{|x| \leq r\}) < \infty \text{ for all } r \in \mathbb{R}\}. \quad (1.3)$$

We regard  $\Theta$  as a complete, separable metric space with the vague topology.

In [11] Spohn constructed an unlabeled dynamics (1.2) in the sense of a Markovian semigroup on  $L^2(\Theta, \mu)$ . Here  $\mu$  is a probability measure on  $(\Theta, \mathfrak{B}(\Theta))$  whose correlation functions are generated by the sine kernel

$$\mathsf{K}_{\sin}(x) = \frac{\bar{\rho}}{\pi x} \sin(\pi x). \quad (1.4)$$

(See Section 2). Here  $0 < \bar{\rho} \leq 1$  is a constant related to the *density* of the particle. Spohn indeed proved the closability of a non-negative bilinear form  $(\mathcal{E}, \mathcal{D}_{\infty})$  on  $L^2(\Theta, \mu)$

$$\begin{aligned} \mathcal{E}(\mathfrak{f}, \mathfrak{g}) &= \int_{\Theta} \mathbb{D}[\mathfrak{f}, \mathfrak{g}](\theta) d\mu, \\ \mathcal{D}_{\infty} &= \{\mathfrak{f} \in \mathcal{D}_{\infty}^{loc} \cap L^2(\Theta, \mu); \mathcal{E}(\mathfrak{f}, \mathfrak{f}) < \infty\}. \end{aligned} \quad (1.5)$$

Here  $\mathbb{D}$  is the square field given by (2.8) and  $\mathcal{D}_{\infty}^{loc}$  is the set of the local smooth functions on  $\Theta$  (see Section 3 for the definition). The Markovian semi-group is given by the Dirichlet form that is the closure  $(\mathcal{E}, \mathcal{D})$  of this closable form on  $L^2(\Theta, \mu)$ .

The measure  $\mu$  is an equilibrium state of (1.2), whose formal Hamiltonian  $\mathcal{H} = \mathcal{H}(\theta)$  is given by  $(\theta = \sum_i \delta_{x_i})$

$$\mathcal{H}(\theta) = \sum_{i \neq j} -2 \log |x_i - x_j|, \quad (1.6)$$

which is a reason we regard Spohn's Markovian semi-group is a correspondent to the dynamics formally given by the SDE (1.1) and (1.2).

We remark the existence of an  $L^2$ -Markovian semigroup does not imply the existence of the associated *diffusion* in general. Here a diffusion means (a family of distributions of) a strong Markov process with continuous sample paths starting from each  $\theta \in \Theta$ .

In [5] it was proved that there exists a diffusion  $(\{\mathsf{P}_{\theta}\}_{\theta \in \Theta}, \{\mathbb{X}_t\})$  with state space  $\Theta$  associated with the Markovian semigroup above. This construction admits us to investigate the *trajectory-wise* properties of the dynamics. In the

present paper we concentrate on the collision property of the diffusion. The problem we are interested in is the following:

Does a pair of particles  $(X_t^i, X_t^j)$  that collides each other for some time  $0 < t < \infty$  exist ?

We say for a diffusion on  $\Theta$  *the non-collision occurs* if the above property does *not* hold, and *the collision occurs* if otherwise.

If the number of particles is finite, then the non-collision should occur at least intuitive level. This is because drifts  $\frac{1}{x_i - x_j}$  have a strong repulsive effect. When the number of the particles is infinite, the non-collision property is non-trivial because the interaction potential is long range and un-integrable. We will prove the non-collision property holds for Dyson's model in infinite dimension.

Since the sine kernel measure is the prototype of determinantal random point fields, it is natural to ask such a non-collision property is universal for stochastic dynamics given by Dirichlet forms (1.5) with the replacement of the measure  $\mu$  with general determinantal random point fields. We will prove, if the kernel of the determinantal random point field (see (2.3)) is locally Lipschitz continuous, then the non-collision always occurs. In addition, we give an example of determinantal random point fields with Hölder continuous kernel that the collision occurs.

The second problem we are interested in this paper is the following:

Does there exist  $\Theta$ -valued diffusions associated with the Dirichlet forms  $(\mathcal{E}, \mathcal{D})$  on  $L^2(\Theta, \mu)$  when  $\mu$  is determinantal random point fields ?

We give a partial answer for this in Theorem 2.5.

The organization of the paper is as follows: In Section 2 we state main theorems. In Section 3 we prepare some notion on configuration spaces. In Section 4 we prove Theorem 2.2 and Theorem 2.3. In Section 5 we prove Proposition 2.9 and Theorem 2.4. In Section 6 we prove Theorem 2.5. Our method proving Theorem 2.1 can be applied to Gibbs measures. So we prove the non-collision property for Gibbs measures in Section 7.

## 2 Set up and the main result

Let  $E \subset \mathbb{R}^d$  be a closed set which is the closure of a connected open set in  $\mathbb{R}^d$  with smooth boundary. Although we will mainly treat the case  $E = \mathbb{R}$ , we give a general framework here by following the line of [10]. Let  $\Theta$  denote the set of configurations on  $E$ , which is defined similarly as (1.3) by replacing  $\mathbb{R}$  with  $E$ .

A probability measure on  $(\Theta, \mathcal{B}(\Theta))$  is called a random point field on  $E$ . Let  $\mu$  be a random point field on  $E$ . A non-negative, permutation invariant function  $\rho_n : E^n \rightarrow \mathbb{R}$  is called an  $n$ -correlation function of  $\mu$  if for any measurable sets  $\{A_1, \dots, A_m\}$  and natural numbers  $\{k_1, \dots, k_m\}$  such that  $k_1 + \dots + k_m = n$

the following holds:

$$\int_{A_1^{k_1} \times \dots \times A_m^{k_m}} \rho_n(x_1, \dots, x_n) dx_1 \cdots dx_n = \int_{\Theta} \prod_{i=1}^m \frac{\theta(A_i)!}{(\theta(A_i) - k_i)!} d\mu.$$

It is known ([10], [3], [4]) that, if a family of non-negative, permutation invariant functions  $\{\rho_n\}$  satisfies

$$\sum_{k=1}^{\infty} \left\{ \frac{1}{(k+j)!} \int_{A^{k+j}} \rho_{k+j} dx_1 \cdots dx_{k+j} \right\}^{-1/k} = \infty, \quad (2.1)$$

then there exists a unique probability measure (random point field)  $\mu$  on  $\mathbb{E}$  whose correlation functions equal  $\{\rho_n\}$ .

Let  $K : L^2(\mathbb{E}, dx) \rightarrow L^2(\mathbb{E}, dx)$  be a non-negative definite operator which is locally trace class; namely

$$\begin{aligned} 0 &\leq (Kf, f)_{L^2(\mathbb{E}, dx)}, \\ \text{Tr}(1_B K 1_B) &< \infty \quad \text{for all bounded Borel set } B. \end{aligned} \quad (2.2)$$

We assume  $K$  has a continuous kernel denoted by  $\mathsf{K} = \mathsf{K}(x, y)$ . Without this assumption one can develop a theory of determinantal random point fields (see [10], [9]); we assume this for the sake of simplicity.

**Definition 2.1.** A probability measure  $\mu$  on  $\Theta$  is said to be a determinantal (or fermion) random point field with kernel  $\mathsf{K}$  if its correlation functions  $\rho_n$  are given by

$$\rho_n(x_1, \dots, x_n) = \det(\mathsf{K}(x_i, x_j)_{1 \leq i, j \leq n}) \quad (2.3)$$

We quote:

**Lemma 2.2** (Theorem 3 in [10]). *Assume  $\mathsf{K}(x, y) = \overline{\mathsf{K}(y, x)}$  and  $0 \leq K \leq 1$ . Then  $K$  determines a unique determinantal random point field  $\mu$ .*

We give examples of determinantal random point fields. The first example is the stationary measure of Dyson's model in infinite dimension. The first three examples are related to the semicircle law of empirical distribution of eigen values of random matrices. We refer to [10] for detail.

**Example 2.3** (sine kernel). Let  $\mathsf{K}_{\text{sin}}$  and  $\bar{\rho}$  be as in (1.4). Then

$$\mathsf{K}_{\text{sin}}(t) = \frac{1}{2\pi} \int_{|k| \leq \pi \bar{\rho}} e^{\sqrt{-1}kt} dk. \quad (2.4)$$

So the  $\mathsf{K}_{\text{sin}}$  is a function of positive type and satisfies the assumptions in Lemma 2.2. Let  $\hat{\mu}^N$  denote the probability measure on  $\mathbb{R}^N$  defined by

$$\hat{\mu}^N = \frac{1}{Z^N} e^{-\sum_{i,j=1}^N -2 \log |x_i - x_j|} e^{-\lambda_N^2 \sum_{i=1}^N x_i^2} dx_1 \cdots dx_N, \quad (2.5)$$

where  $\lambda_N = 2(\pi\bar{\rho})^3/3N^2$  and  $Z^N$  is the normalization. Set  $\mu^N = \hat{\mu}^N \circ (\xi^N)^{-1}$ , where  $\xi^N : \mathbb{R}^N \rightarrow \Theta$  such that  $\xi^N(x_1, \dots, x_N) = \sum_{i=1}^N \delta_{x_i}$ . Let  $\rho_n^N$  denote the  $n$ -correlation function of  $\mu^N$ . Let  $\rho_n$  denote the  $n$ -correlation function of  $\mu$ . Then it is known ([11, Proposition 1], [10]) that for all  $n = 1, 2, \dots$

$$\lim_{N \rightarrow \infty} \rho_n^N(x_1, \dots, x_n) = \rho_n(x_1, \dots, x_n) \quad \text{for all } (x_1, \dots, x_n). \quad (2.6)$$

In this sense the measure  $\mu$  is associated with the Hamiltonian  $\mathcal{H}$  in (1.6) coming from the log potential  $-2 \log|x|$ .

**Example 2.4** (Airy kernel).  $E = \mathbb{R}$  and

$$K(x, y) = \frac{\mathcal{A}_i(x) \cdot \mathcal{A}'_i(y) - \mathcal{A}_i(y) \cdot \mathcal{A}'_i(x)}{x - y}$$

Here  $\mathcal{A}_i$  is the Airy function.

**Example 2.5** (Bessel kernel). Let  $E = [0, \infty)$  and

$$K(x, y) = \frac{J_\alpha(\sqrt{x}) \cdot \sqrt{y} \cdot J'_\alpha(\sqrt{y}) - J_\alpha(\sqrt{y}) \cdot \sqrt{x} \cdot J'_\alpha(\sqrt{x})}{2(x - y)}.$$

Here  $J_\alpha$  is the Bessel function of order  $\alpha$ .

**Example 2.6.** Let  $E = \mathbb{R}$  and  $K(x, y) = m(x)k(x - y)m(y)$ , where  $k : \mathbb{R} \rightarrow \mathbb{R}$  is a non-negative, continuous *even* function that is convex in  $[0, \infty)$  such that  $k(0) \leq 1$ , and  $m : \mathbb{R} \rightarrow \mathbb{R}$  is nonnegative continuous and  $\int_{\mathbb{R}} m(t)dt < \infty$  and  $m(x) \leq 1$  for all  $x$  and  $0 < m(x)$  for some  $x$ . Then  $K$  satisfies the assumptions in Lemma 2.2. Indeed, it is well-known that  $k$  is a function of positive type (187 p. in [1] for example), so the Fourier transformation of a finite positive measure. By assumption  $0 \leq K(x, y) \leq 1$ , which implies  $0 \leq K \leq 1$ . Since  $\int K(x, x)dx < \infty$ ,  $K$  is of trace class.

Let  $A$  denote the subset of  $\Theta$  defined by

$$A = \{\theta \in \Theta; \theta(\{x\}) \geq 2 \quad \text{for some } x \in E\}. \quad (2.7)$$

Note that  $A$  denotes the set consisting of the configurations with collisions. We are interested in how large the set  $A$  is. Of course  $\mu(A) = 0$  because the 2-correlation function is locally integrable. We study  $A$  more closely from the point of stochastic dynamics; namely, we measure  $A$  by using a capacity.

To introduce the capacity we next consider a bilinear form related to the given probability measure  $\mu$ . Let  $\mathcal{D}_\infty^{loc}$  be the set of all local, smooth functions on  $\Theta$  defined in Section 3. For  $f, g \in \mathcal{D}_\infty^{loc}$  we set  $\mathbb{D}[f, g] : \Theta \rightarrow \mathbb{R}$  by

$$\mathbb{D}[f, g](\theta) = \frac{1}{2} \sum_i \frac{\partial f(\mathbf{x})}{\partial x_i} \frac{\partial g(\mathbf{x})}{\partial x_i}. \quad (2.8)$$

Here  $\theta = \sum_i \delta_{x_i}$ ,  $\mathbf{x} = (x_1, \dots)$  and  $f(\mathbf{x}) = f(x_1, \dots)$  is the permutation invariant function such that  $f(\theta) = f(x_1, x_2, \dots)$  for all  $\theta \in \Theta$ . We set  $g$  similarly. Note

that the left hand side of (2.8) is again permutation invariant. Hence it can be regarded as a function of  $\theta = \sum_i \delta_{x_i}$ . Such  $f$  and  $g$  are unique; so the function  $\mathbb{D}[f, g]: \Theta \rightarrow \mathbb{R}$  is well defined.

For a probability measure  $\mu$  in  $\Theta$  we set as before

$$\mathcal{E}(f, g) = \int_{\Theta} \mathbb{D}[f, g](\theta) d\mu,$$

$$\mathcal{D}_{\infty} = \{f \in \mathcal{D}_{\infty}^{loc} \cap L^2(\Theta, \mu); \mathcal{E}(f, f) < \infty\}.$$

When  $(\mathcal{E}, \mathcal{D}_{\infty})$  is closable on  $L^2(\Theta, \mu)$ , we denote its closure by  $(\mathcal{E}, \mathcal{D})$ .

We are now ready to introduce a notion of capacity for a pre-Dirichlet space  $(\mathcal{E}, \mathcal{D}_{\infty}, L^2(\Theta, \mu))$ . Let  $\mathcal{O}$  denote the set consisting of all open sets in  $\Theta$ . For  $O \in \mathcal{O}$  we set  $\mathcal{L}_O = \{f \in \mathcal{D}_{\infty}; f \geq 1 \text{ } \mu\text{-a.e. on } O\}$  and

$$\text{Cap}(O) = \begin{cases} \inf_{f \in \mathcal{L}_O} \{\mathcal{E}(f, f) + (f, f)_{L^2(\Theta, \mu)}\} & \mathcal{L}_O \neq \emptyset \\ \infty & \mathcal{L}_O = \emptyset \end{cases}.$$

For an arbitrary subset  $A \subset \Theta$  we set  $\text{Cap}(A) = \inf_{A \subset O \in \mathcal{O}} \text{Cap}(O)$ . This quantity  $\text{Cap}$  is called 1-capacity for the pre-Dirichlet space  $(\mathcal{E}, \mathcal{D}_{\infty}, L^2(\Theta, \mu))$ .

We state the main theorem:

**Theorem 2.1.** *Let  $\mu$  be a determinantal random point field with kernel  $K$ . Assume  $K$  is locally Lipschitz continuous. Then*

$$\text{Cap}(A) = 0, \tag{2.9}$$

where  $A$  is given by (2.7).

In [5] it was proved

**Lemma 2.7** (Corollary 1 in [5]). *Let  $\mu$  be a probability measure on  $\Theta$ . Assume  $\mu$  has locally bounded correlation functions. Assume  $(\mathcal{E}, \mathcal{D}_{\infty})$  is closable on  $L^2(\Theta, \mu)$ . Then there exists a diffusion  $(\{P_{\theta}\}_{\theta \in \Theta}, \{\mathbb{X}_t\})$  associated with the Dirichlet space  $(\mathcal{E}, \mathcal{D}, L^2(\Theta, \mu))$ .*

Combining this with Theorem 2.1 we have

**Theorem 2.2.** *Assume  $\mu$  satisfies the assumption in Theorem 2.1. Assume  $(\mathcal{E}, \mathcal{D}_{\infty})$  is closable on  $L^2(\Theta, \mu)$ . Then a diffusion  $(\{P_{\theta}\}_{\theta \in \Theta}, \{\mathbb{X}_t\})$  associated with the Dirichlet space  $(\mathcal{E}, \mathcal{D}, L^2(\Theta, \mu))$  exists and satisfies*

$$P_{\theta}(\sigma_A = \infty) = 1 \quad \text{for q.e. } \theta, \tag{2.10}$$

where  $\sigma_A = \inf\{t > 0; \mathbb{X}_t \in A\}$ .

We refer to [2] for q.e. (quasi everywhere) and related notions on Dirichlet form theory. We remark the capacity of pre-Dirichlet forms are bigger than or equal to the one of its closure by definition. So (2.10) is an immediate consequence of Theorem 2.1 and the general theory of Dirichlet forms once  $(\mathcal{E}, \mathcal{D}_{\infty})$  is closable on  $L^2(\Theta, \mu)$  and the resulting (quasi) regular Dirichlet space  $(\mathcal{E}, \mathcal{D}, L^2(\Theta, \mu))$  exists.

To apply Theorem 2.2 to Dyson's model we recall a result of Spohn.

**Lemma 2.8** (Proposition 4 in [11]). *Let  $\mu$  be the determinantal random point field with the sine kernel in Example 2.3. Then  $(\mathcal{E}, \mathcal{D}_\infty)$  is closable on  $L^2(\Theta, \mu)$ .*

We say a diffusion  $(\{P_\theta\}_{\theta \in \Theta}, \{\mathbb{X}_t\})$  is Dyson's model in infinite dimension if it is associated with the Dirichlet space  $(\mathcal{E}, \mathcal{D}, L^2(\Theta, \mu))$  in Theorem 2.8. Collecting these we conclude:

**Theorem 2.3.** *No collision (2.10) occurs in Dyson's model in infinite dimension.*

The assumption of the local Lipschitz continuity of the kernel  $K$  is crucial; we next give a collision example when  $K$  is merely Hölder continuous. We prepare:

**Proposition 2.9.** *Assume  $K$  is of trace class. Then  $(\mathcal{E}, \mathcal{D}_\infty)$  is closable on  $L^2(\Theta, \mu)$ .*

**Theorem 2.4.** *Let  $K(x, y) = m(x)k(x - y)m(y)$  be as in Example 2.6. Let  $\alpha$  be a constant such that*

$$0 < \alpha < 1. \quad (2.11)$$

*Assume  $m$  and  $k$  are continuous and there exist positive constants  $c_1$  and  $c_2$  such that*

$$c_1 t^\alpha \leq k(0) - k(t) \leq c_2 t^\alpha \quad \text{for } 0 \leq t \leq 1. \quad (2.12)$$

*Then  $(\mathcal{E}, \mathcal{D}_\infty, L^2(\Theta, \mu))$  is closable and the associated diffusion satisfies*

$$P_\theta(\sigma_A < \infty) = 1 \quad \text{for q.e. } \theta. \quad (2.13)$$

Unfortunately the closability of the pre-Dirichlet form  $(\mathcal{E}, \mathcal{D}_\infty)$  on  $L^2(\Theta, \mu)$  has not yet proved for determinantal random point fields of locally trace class except the sine kernel. So we propose a problem:

**Problem 2.10.** (1) Are pre-Dirichlet forms  $(\mathcal{E}, \mathcal{D}_\infty)$  on  $L^2(\Theta, \mu)$  closable when  $\mu$  are determinantal random fields with continuous kernels?  
(2) Can one construct stochastic dynamics (diffusion processes) associated with pre-Dirichlet forms  $(\mathcal{E}, \mathcal{D}_\infty)$  on  $L^2(\Theta, \mu)$ .

We remark one can deduce the second problem from the first one (see [5, Theorem 1]). We conjecture that  $(\mathcal{E}, \mathcal{D}_\infty, L^2(\Theta, \mu))$  are always closable. As we see above, in case of trace class kernel, this problem is solved by Proposition 2.9. But it is important to prove this for determinantal random point field of *locally* trace class. This class contains Airy kernel and Bessel kernel and other nutritious examples. We also remark for interacting Brownian motions with Gibbsian equilibriums this problem was settled successfully ([5]).

In the next theorem we give a partial answer for (2) of Problem 2.10. We will show one can construct a stochastic dynamics in infinite volume, which is canonical in the sense that (1) it is the strong resolvent limit of a sequence of

finite volume dynamics and that (2) it coincides with  $(\mathcal{E}, \mathcal{D})$  whenever  $(\mathcal{E}, \mathcal{D}_\infty)$  is closable on  $L^2(\Theta, \mu)$ .

For two symmetric, nonnegative forms  $(\mathcal{E}_1, \mathcal{D}_1)$  and  $(\mathcal{E}_2, \mathcal{D}_2)$ , we write  $(\mathcal{E}_1, \mathcal{D}_1) \leq (\mathcal{E}_2, \mathcal{D}_2)$  if  $\mathcal{D}_1 \supset \mathcal{D}_2$  and  $\mathcal{E}_1(f, f) \leq \mathcal{E}_2(f, f)$  for all  $f \in \mathcal{D}_2$ . Let  $(\mathcal{E}^{\text{reg}}, \mathcal{D}^{\text{reg}})$  denote the regular part of  $(\mathcal{E}, \mathcal{D}_\infty)$  on  $L^2(\Theta, \mu)$ , that is,  $(\mathcal{E}^{\text{reg}}, \mathcal{D}^{\text{reg}})$  is closable on  $L^2(\Theta, \mu)$  and in addition satisfies the following:

$$(\mathcal{E}^{\text{reg}}, \mathcal{D}^{\text{reg}}) \leq (\mathcal{E}, \mathcal{D}_\infty),$$

and for all closable forms such that  $(\mathcal{E}', \mathcal{D}') \leq (\mathcal{E}, \mathcal{D}_\infty)$

$$(\mathcal{E}', \mathcal{D}') \leq (\mathcal{E}^{\text{reg}}, \mathcal{D}^{\text{reg}}).$$

It is well known that such a  $(\mathcal{E}^{\text{reg}}, \mathcal{D}^{\text{reg}})$  exists uniquely and called the maximal regular part of  $(\mathcal{E}, \mathcal{D})$ . Let us denote the closure by the same symbol  $(\mathcal{E}^{\text{reg}}, \mathcal{D}^{\text{reg}})$ .

Let  $\pi_r: \Theta \rightarrow \Theta$  be such that  $\pi_r(\theta) = \theta(\cdot \cap \{x \in \mathbb{E}; |x| < r\})$ . We set

$$\mathcal{D}_{\infty, r} = \{f \in \mathcal{D}_\infty; f \text{ is } \sigma[\pi_r]\text{-measurable}\}.$$

We will prove  $(\mathcal{E}, \mathcal{D}_{\infty, r})$  are closable on  $L^2(\Theta, \mu)$ . These are the finite volume dynamics we are considering.

Let  $\mathbb{G}_\alpha$  (resp.  $\mathbb{G}_{r, \alpha}$ ) ( $\alpha > 0$ ) denote the  $\alpha$ -resolvent of the semi-group associated with the closure of  $(\mathcal{E}^{\text{reg}}, \mathcal{D}^{\text{reg}})$  (resp.  $(\mathcal{E}, \mathcal{D}_{\infty, r})$ ) on  $L^2(\Theta, \mu)$ .

**Theorem 2.5.** (1)  $(\mathcal{E}^{\text{reg}}, \mathcal{D}^{\text{reg}})$  on  $L^2(\Theta, \mu)$  is a quasi-regular Dirichlet form. So the associated diffusion exists.  
(2)  $\mathbb{G}_{r, \alpha}$  converge to  $\mathbb{G}_\alpha$  strongly in  $L^2(\Theta, \mu)$  for all  $\alpha > 0$ .

*Remark 2.11.* We think the diffusion constructed in Theorem 2.5 is a reasonable one because of the following reason. (1) By definition the closure of  $(\mathcal{E}^{\text{reg}}, \mathcal{D}^{\text{reg}})$  equals  $(\mathcal{E}, \mathcal{D})$  when  $(\mathcal{E}, \mathcal{D}_\infty)$  is closable. (2) One naturally associated Markov processes on  $\Theta_r$ , where  $\Theta_r$  is the set of configurations on  $\mathbb{E} \cap \{|x| < r\}$ . So (2) of Theorem 2.5 implies the diffusion is the strong resolvent limit of finite volume dynamics.

*Remark 2.12.* If one replace  $\mu$  by the Poisson random measure  $\lambda$  whose intensity measure is the Lebesgue measure and consider the Dirichlet space  $(\mathcal{E}^\lambda, \mathcal{D})$  on  $L(\Theta, \lambda)$ , then the associated  $\Theta$ -valued diffusion is the  $\Theta$ -valued Brownian motion  $\mathbb{B}$ , that is, it is given by

$$\mathbb{B}_t = \sum_{i=1}^{\infty} \delta_{B_t^i},$$

where  $\{B_t^i\}$  ( $i \in \mathbb{N}$ ) are infinite amount of independent Brownian motions. In this sense we say in Abstract that the Dirichlet form given by (1.5) for Radon measures in  $\Theta$  *canonical*. We also remark such a type of local Dirichlet forms are often called distorted Brownian motions.



### 3 Preliminary

Let  $I_r = (-r, r)^d \cap \mathbf{E}$  and  $\Theta_r^n = \{\theta \in \Theta; \theta(I_r) = n\}$ . We note  $\Theta = \sum_{n=0}^{\infty} \Theta_r^n$ . Let  $I_r^n$  be the  $n$  times product of  $I_r$ . We define  $\pi_r: \Theta \rightarrow \Theta$  by  $\pi_r(\theta) = \theta(\cdot \cap I_r)$ . A function  $\mathbf{x}: \Theta_r^n \rightarrow I_r^n$  is called a  $I_r^n$ -coordinate of  $\theta$  if

$$\pi_r(\theta) = \sum_{k=1}^n \delta_{x_k(\theta)}, \quad \mathbf{x}(\theta) = (x_1(\theta), \dots, x_n(\theta)). \quad (3.1)$$

Suppose  $\mathbf{f}: \Theta \rightarrow \mathbb{R}$  is  $\sigma[\pi_r]$ -measurable. Then for each  $n = 1, 2, \dots$  there exists a unique permutation invariant function  $f_r^n: I_r^n \rightarrow \mathbb{R}$  such that

$$\mathbf{f}(\theta) = f_r^n(\mathbf{x}(\theta)) \quad \text{for all } \theta \in \Theta_r^n. \quad (3.2)$$

We next introduce mollifier. Let  $j: \mathbb{R} \rightarrow \mathbb{R}$  be a non-negative, smooth function such that  $j(x) = j(|x|)$ ,  $\int_{\mathbb{R}^d} j dx = 1$  and  $j(x) = 0$  for  $|x| \geq \frac{1}{2}$ . Let  $j_\epsilon = \epsilon j(\cdot/\epsilon)$  and  $j_\epsilon^n(x_1, \dots, x_n) = \prod_{i=1}^n j_\epsilon(x_i)$ . For a  $\sigma[\pi_r]$ -measurable function  $\mathbf{f}$  we set  $\mathfrak{J}_{r,\epsilon}\mathbf{f}: \Theta \rightarrow \mathbb{R}$  by

$$\mathfrak{J}_{r,\epsilon}\mathbf{f}(\theta) = \begin{cases} j_\epsilon^n * \hat{f}_r^n(\mathbf{x}(\theta)) & \text{for } \theta \in \Theta_r^n, n \geq 1 \\ \mathbf{f}(\theta) & \text{for } \theta \in \Theta_r^0, \end{cases} \quad (3.3)$$

where  $f_r^n$  is given by (3.2) for  $\mathbf{f}$ , and  $\hat{f}_r^n: \mathbb{R}^{dn} \rightarrow \mathbb{R}$  is the function defined by  $\hat{f}_r^n(x) = f_r^n(x)$  for  $x \in I_r^n$  and  $\hat{f}_r^n(x) = 0$  for  $x \notin I_r^n$ . Moreover  $\mathbf{x}(\theta)$  is an  $I_r^n$ -coordinate of  $\theta \in \Theta_r^n$ , and  $*$  denotes the convolution in  $\mathbb{R}^n$ . It is clear that  $\mathfrak{J}_{r,\epsilon}\mathbf{f}$  is  $\sigma[\pi_r]$ -measurable.

We say a function  $\mathbf{f}: \Theta \rightarrow \mathbb{R}$  is local if  $\mathbf{f}$  is  $\sigma[\pi_r]$ -measurable for some  $r < \infty$ . For  $\mathbf{f}: \Theta \rightarrow \mathbb{R}$  and  $n \in \mathbb{N} \cup \{\infty\}$  there exists a unique permutation function  $f^n$  such that  $\mathbf{f}(\theta) = f^n(x_1, \dots)$  for all  $\theta \in \Theta^n$ . Here  $\Theta^n = \{\theta \in \Theta; \theta(\mathbf{E}) = n\}$ , and  $\theta = \sum_i \delta_{x_i}$ . A function  $\mathbf{f}$  is called smooth if  $f^n$  is smooth for all  $n \in \mathbb{N} \cup \{\infty\}$ . Note that a  $\sigma[\pi_r]$ -measurable function  $\mathbf{f}$  is smooth if and only if  $f_r^n$  is smooth for all  $n \in \mathbb{N}$ .

### 4 Proof of Theorem 2.2

We give a sequence of reductions of (2.9). Let  $\mathbf{A}$  denote the set consisting of the sequences  $\mathbf{a} = (a_r)_{r \in \mathbb{N}}$  satisfying the following:

$$a_r \in \mathbb{Q} \quad \text{for all } r \in \mathbb{N}, \quad (4.1)$$

$$a_r = 2r + r_0 \quad \text{for all sufficiently large } r \in \mathbb{N}, \quad (4.2)$$

$$2 \leq a_1, \quad 1 \leq a_{r+1} - a_r \leq 2 \quad \text{for all } r \in \mathbb{N}. \quad (4.3)$$

Note that the cardinality of  $\mathbf{A}$  is countable by (4.1) and (4.2).

Let  $\mathbb{I} = \{2, 3, \dots\}^3$ . For  $(r, n, m) \in \mathbb{I}$  and  $\mathbf{a} = (a_r) \in \mathbf{A}$  we set

$$\Theta^{\mathbf{a}}(r, n) = \{\theta \in \Theta; \theta(I_{a_r}) = n\}$$

$$\Theta^{\mathbf{a}}(r, n, m) = \{\theta \in \Theta; \theta(I_{a_r}) = n, \theta(\bar{I}_{a_r + \frac{1}{m}} \setminus I_{a_r}) = 0\}.$$

Here  $\bar{I}_{a_r+\frac{1}{m}}$  is the closure of  $I_{a_r+\frac{1}{m}}$ , where  $I_r = (-r, r)^d \cap \mathbb{E}$  as before. We remark  $\Theta^{\mathbf{a}}(r, n, m)$  is an open set in  $\Theta$ . We set

$$\begin{aligned} \mathbf{A}_\epsilon^{\mathbf{a}}(r, n, m) = \{ \theta = \sum_i \delta_{x_i} ; \theta \in \Theta^{\mathbf{a}}(r, n, m) \text{ and } \theta \text{ satisfy} \\ |x_i - x_j| < \epsilon \text{ and } x_i, x_j \in I_{a_r-1} \text{ for some } i \neq j \}. \end{aligned} \quad (4.4)$$

It is clear that  $\mathbf{A}_\epsilon^{\mathbf{a}}(r, n, m)$  is an open set in  $\Theta$ .

**Lemma 4.1.** *Assume that for all  $\mathbf{a} \in \mathbf{A}$  and  $(r, n, m) \in \mathbb{I}$*

$$\inf_{0 < \epsilon < 1/2m} \text{Cap}(\mathbf{A}_\epsilon^{\mathbf{a}}(r, n, m)) = 0. \quad (4.5)$$

*Then (2.9) holds.*

*Proof.* Let

$$\begin{aligned} \mathbf{A}^{\mathbf{a}}(r, n, m) = \{ \theta = \sum_i \delta_{x_i} ; \theta \in \Theta^{\mathbf{a}}(r, n, m) \text{ and } \theta \text{ satisfy} \\ x_i = x_j \text{ and } x_i, x_j \in I_{a_r-1} \text{ for some } i \neq j \}. \end{aligned}$$

Then  $\mathbf{A} = \bigcup_{\mathbf{a} \in \mathbf{A}} \bigcup_{(r, n, m) \in \mathbb{I}} \mathbf{A}^{\mathbf{a}}(r, n, m)$ . Since  $\mathbf{A}$  and  $\mathbb{I}$  are countable sets and the capacity is sub additive, (2.9) follows from

$$\text{Cap}(\mathbf{A}^{\mathbf{a}}(r, n, m)) = 0 \quad \text{for all } \mathbf{a} \in \mathbf{A}, (r, n, m) \in \mathbb{I}. \quad (4.6)$$

Note that  $\mathbf{A}^{\mathbf{a}}(r, n, m) \subset \mathbf{A}_\epsilon^{\mathbf{a}}(r, n, m)$ . So (4.5) implies (4.6) by the monotonicity of the capacity, which deduces (2.9).  $\square$

Now fix  $\mathbf{a} \in \mathbf{A}$  and  $(r, n, m) \in \mathbb{I}$  and suppress them from the notion. Set

$$\mathbf{A}_\epsilon^- = \mathbf{A}_{\epsilon/2}^{\mathbf{a}}(r, n, m), \quad \mathbf{A}_\epsilon = \mathbf{A}_\epsilon^{\mathbf{a}}(r, n, m), \quad \mathbf{A}_\epsilon^+ = \mathbf{A}_{1+\epsilon}^{\mathbf{a}}(r, n, m). \quad (4.7)$$

and let  $h_\epsilon : \mathbb{R} \rightarrow \mathbb{R}$  ( $0 < \epsilon < 1/m < 1$ ) such that

$$h_\epsilon(t) = \begin{cases} 2 & (|t| \leq \epsilon) \\ 2 \log |t| / \log \epsilon & (\epsilon \leq |t| \leq 1) \\ 0 & (1 \leq |t|). \end{cases} \quad (4.8)$$

We define  $\mathfrak{h}_\epsilon : \Theta \rightarrow \mathbb{R}$  by  $\mathfrak{h}_\epsilon(\theta) = 0$  for  $\theta \notin \Theta^{\mathbf{a}}(r, n, m)$  and

$$\mathfrak{h}_\epsilon(\theta) = \sum_{x_i, x_j \in I_{a_r-1}, j \neq i} h_\epsilon(x_i - x_j) \quad \text{for } \theta \in \Theta^{\mathbf{a}}(r, n, m).$$

Here we set  $\mathfrak{h}_\epsilon(\theta) = 0$  if the summand is empty. Let  $\mathfrak{g}_\epsilon = \mathfrak{J}_{a_r+\frac{1}{m}, \epsilon/4} \mathfrak{h}_\epsilon$ . Here  $\mathfrak{J}_{a_r+\frac{1}{m}, \epsilon/4}$  is the mollifier introduced in (3.3).

**Lemma 4.2.** For  $0 < \epsilon < 1/2m$ ,  $\mathbf{g}_\epsilon$  satisfy the following:

$$\mathbf{g}_\epsilon \in \mathcal{D}_\infty \quad (4.9)$$

$$\mathbf{g}_\epsilon(\theta) \geq 1 \quad \text{for all } \theta \in \mathbf{A}_\epsilon \quad (4.10)$$

$$0 \leq \mathbf{g}_\epsilon(\theta) \leq n(n+1) \quad \text{for all } \theta \in \Theta \quad (4.11)$$

$$\mathbf{g}_\epsilon(\theta) = 0 \quad \text{for all } \theta \notin \mathbf{A}_\epsilon^+ \quad (4.12)$$

$$\mathbb{D}[\mathbf{g}_\epsilon, \mathbf{g}_\epsilon](\theta) = 0 \quad \text{for all } \theta \notin \mathbf{A}_\epsilon^+ \setminus \mathbf{A}_\epsilon^- \quad (4.13)$$

$$\mathbb{D}[\mathbf{g}_\epsilon, \mathbf{g}_\epsilon](\theta) \leq \frac{c_3}{(\log \epsilon \min |x_i - x_j|)^2} \quad \text{for all } \theta \in \mathbf{A}_\epsilon^+ \setminus \mathbf{A}_\epsilon^-. \quad (4.14)$$

Here  $\theta = \sum \delta_{x_k}$  and the minimum in (4.14) is taken over  $x_i, x_j$  such that

$$x_i, x_j \in I_{a_r-1}, \quad \epsilon/2 \leq |x_i - x_j| \leq 1 + \epsilon,$$

and  $c_3 \geq 0$  is a constant independent of  $\epsilon$  ( $c_3$  depends on  $(r, n, m)$ ).

*Proof.* (4.9) follows from [5, Lemma 2.4 (1)]. Other statements are clear from a direct calculation.  $\square$

Permutation invariant functions  $\sigma_r^n : I_r^n \rightarrow \mathbb{R}^+$  are called density functions of  $\mu$  if, for all bounded  $\sigma[\pi_r]$ -measurable functions  $\mathfrak{f}$ ,

$$\int_{\Theta_r^n} \mathfrak{f} d\mu = \frac{1}{n!} \int_{I_r^n} f_r^n \sigma_r^n dx. \quad (4.15)$$

Here  $f_r^n : I_r^n \rightarrow \mathbb{R}$  is the permutation invariant function such that  $f_r^n(\mathbf{x}(\theta)) = \mathfrak{f}(\theta)$  for  $\theta \in \Theta_r^n$ , where  $\mathbf{x}$  is an  $I_r^n$ -coordinate. We recall relations between a correlation function and a density function ([10]):

$$\rho_n = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{I_r^k} \sigma_r^{n+k}(x_1, \dots, x_{n+k}) dx_{n+1} \cdots dx_{n+k} \quad (4.16)$$

$$\sigma_r^n = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{I_r^k} \rho_{n+k}(x_1, \dots, x_{n+k}) dx_{n+1} \cdots dx_{n+k} \quad (4.17)$$

The first summand in the right hand side of (4.16) is taken to be  $\sigma_r^n$ . It is clear that

$$0 \leq \sigma_r^n(x_1, \dots, x_n) \leq \rho_n(x_1, \dots, x_n) \quad (4.18)$$

**Lemma 4.3.** There exists a constant  $c_4$  depending on  $r, n$  such that

$$\sigma_r^n(x_1, \dots, x_n) \leq c_4 \min_{i \neq j} |x_i - x_j| \quad \text{for all } (x_1, \dots, x_n) \in I_r^n \quad (4.19)$$

*Proof.* By (2.3) and the kernel  $\mathbf{K}$  is locally Lipschitz continuous, we see  $\rho_n$  is bounded and Lipschitz continuous on  $I_r^n$ . In addition, by using (2.3) we see

$\rho_n = 0$  if  $x_i = x_j$  for some  $i \neq j$ . Hence by using (2.3) again there exists a constant  $c_5$  depending on  $n, r$  such that

$$\rho_n(x_1, \dots, x_n) \leq c_5 \min_{i \neq j} |x_i - x_j| \quad \text{for all } (x_1, \dots, x_n) \in I_r^n. \quad (4.20)$$

(4.19) follows from this and (4.18) immediately.  $\square$

**Lemma 4.4.** (4.5) holds true.

*Proof.* By the definition of the capacity,  $\mathbf{g}_\epsilon \in \mathcal{D}_\infty$ , (4.9) and (4.10) we obtain

$$\text{Cap}(\mathbf{A}_\epsilon) \leq \mathcal{E}(\mathbf{g}_\epsilon, \mathbf{g}_\epsilon) + (\mathbf{g}_\epsilon, \mathbf{g}_\epsilon)_{L^2(\Theta, \mu)} \quad (4.21)$$

So we will estimate the right hand side. We now see by (4.13)

$$\begin{aligned} \mathcal{E}(\mathbf{g}_\epsilon, \mathbf{g}_\epsilon) &= \int_{\mathbf{A}_\epsilon^+ \setminus \mathbf{A}_\epsilon^-} \mathbb{D}[\mathbf{g}_\epsilon, \mathbf{g}_\epsilon] d\mu \\ &= \frac{1}{n!} \int_{B_\epsilon} \left\{ \frac{1}{2} \sum_{i=1}^n \frac{\partial g_\epsilon^n}{\partial x_i} \frac{\partial g_\epsilon^n}{\partial x_i} \right\} \sigma_{a_r + \frac{1}{m}}^n dx_1 \cdots dx_n \\ &=: \mathbf{I}_\epsilon. \end{aligned} \quad (4.22)$$

Here  $g_\epsilon^n$  is defined by (3.2) for  $\mathbf{g}_\epsilon$ , and  $B_\epsilon = \varpi_{a_r + \frac{1}{m}}^{-1}(\pi_{a_r + \frac{1}{m}}(\mathbf{A}_\epsilon^+ \setminus \mathbf{A}_\epsilon^-))$ , where  $\varpi: I_{a_r + \frac{1}{m}}^n \rightarrow \Theta$  is the map such that  $\varpi((x_1, \dots, x_n)) = \sum \delta_{x_i}$ .

By using (4.14) and Lemma 4.3 for  $a_r + \frac{1}{m}$  it is not difficult to see there exists a constant  $c_6$  independent of  $\epsilon$  satisfying the following:

$$\mathbf{I}_\epsilon \leq \frac{c_6}{|\log \epsilon|}.$$

This implies  $\lim_{\epsilon \rightarrow 0} \mathcal{E}(\mathbf{g}_\epsilon, \mathbf{g}_\epsilon) = 0$ . By (4.11) and (4.12) we have

$$(\mathbf{g}_\epsilon, \mathbf{g}_\epsilon)_{L^2(\Theta, \mu)} = \int_{\mathbf{A}_\epsilon^+} \mathbf{g}_\epsilon^2 d\mu \leq n^2(n+1)^2 \mu(\mathbf{A}_\epsilon^+) \rightarrow 0 \quad \text{as } \epsilon \downarrow 0.$$

Combining these with (4.21) we complete the proof of Lemma 4.4.  $\square$

*Proof of Theorem 2.1.* Theorem 2.1 follows from Lemma 4.1 and Lemma 4.4 immediately.  $\square$

## 5 Proof of Proposition 2.9

**Lemma 5.1.** Let  $\mu$  be a probability measure on  $(\Theta, \mathcal{B}(\Theta))$  such that  $\mu(\{\theta(\mathbf{E}) < \infty\}) = 1$  and that density functions  $\{\sigma_{\mathbf{E}}^n\}$  on  $\mathbf{E}$  of  $\mu$  are continuous. Then  $(\mathcal{E}, \mathcal{D}_\infty)$  is closable on  $L^2(\Theta, \mu)$ .

*Proof.* Let  $\Theta^n = \{\theta \in \Theta; \theta(\mathbf{E}) = n\}$  and set

$$\mathcal{E}^n(\mathbf{f}, \mathbf{g}) = \sum_{k=1}^n \int_{\Theta^k} \mathbb{D}[\mathbf{f}, \mathbf{g}] d\mu.$$

By assumption  $\sum_{n=0}^{\infty} \mu(\Theta^n) = 1$ , from which we deduce  $(\mathcal{E}, \mathcal{D}_{\infty})$  is the increasing limit of  $\{(\mathcal{E}^n, \mathcal{D}_{\infty})\}$ . Since density functions are continuous, each  $(\mathcal{E}^n, \mathcal{D}_{\infty})$  is closable on  $L^2(\Theta, \mu)$ . So its increasing limit  $(\mathcal{E}, \mathcal{D}_{\infty})$  is also closable on  $L^2(\Theta, \mu)$ .  $\square$

**Lemma 5.2.** *Let  $\mu$  be a determinantal random point field on  $\mathbf{E}$  with continuous kernel  $\mathbf{K}$ . Assume  $\mathbf{K}$  is of trace class. Then their density functions  $\sigma^n$  on  $\mathbf{E}$  are continuous.*

*Proof.* For the sake of simplicity we only prove the case  $K < 1$ , where  $K$  is the operator generated by the integral kernel  $\mathbf{K}$ . The general case is proved similarly by using a device in [10, 935 p.].

Let  $\lambda_i$  denote the  $i$ -th eigenvalue of  $K$  and  $\varphi_i$  its normalized eigenfunction. Then since  $K$  is of trace class we have

$$\mathbf{K}(x, y) = \sum_{i=1}^{\infty} \lambda_i \varphi_i(x) \overline{\varphi_i(y)}. \quad (5.1)$$

It is known that (see [10, 934 p.])

$$\sigma^n(x_1, \dots, x_n) = \det(\text{Id} - K) \cdot \det(L(x_i, x_j))_{1 \leq i, j \leq n}, \quad (5.2)$$

where  $\det(\text{Id} - K) = \prod_{i=1}^{\infty} (1 - \lambda_i)$  and

$$L(x, y) = \sum_{i=1}^{\infty} \frac{\lambda_i}{1 - \lambda_i} \varphi_i(x) \overline{\varphi_i(y)}. \quad (5.3)$$

Since  $\mathbf{K}(x, y)$  is continuous, eigenfunctions  $\varphi_i(x)$  are also continuous. It is well known that the right hand side of (5.1) converges uniformly. By  $0 \leq K < 1$  we have  $0 \leq \lambda_i \leq \lambda_1 < 1$ . Collecting these implies the right hand side of (5.3) converges uniformly. Hence  $L(x, y)$  is continuous in  $(x, y)$ . This combined with (5.2) completes the proof.  $\square$

*Proof of Proposition 2.9.* Since  $K$  is of trace class, the associated determinantal random point field  $\mu$  satisfies  $\mu(\{\theta(\mathbf{E}) < \infty\}) = 1$ . By Lemma 5.2 we have density functions  $\sigma_{\mathbf{E}}^n$  are continuous. So Proposition 2.9 follows from Lemma 5.1.  $\square$

We now turn to the proof of Theorem 2.4. So as in the statement in Theorem 2.4 let  $\mathbf{E} = \mathbb{R}$  and  $\mathbf{K}(x, y) = \mathbf{m}(x) \mathbf{k}(x - y) \mathbf{m}(y)$ , where  $\mathbf{k}: \mathbb{R} \rightarrow \mathbb{R}$  is a non-negative, continuous *even* function that is convex in  $[0, \infty)$  such that  $\mathbf{k}(0) \leq 1$ , and  $\mathbf{m}: \mathbb{R} \rightarrow \mathbb{R}$  is nonnegative continuous and  $\int_{\mathbb{R}} \mathbf{m}(t) dt < \infty$  and  $\mathbf{m}(x) \leq 1$  for all  $x$  and  $0 < \mathbf{m}(x)$  for some  $x$ . We assume  $\mathbf{k}$  satisfies (2.12).

**Lemma 5.3.** *There exists an interval  $I$  in  $\mathbf{E}$  such that*

$$\sigma_I^2(x, x+t) \geq c_7 t^\alpha \quad \text{for all } |t| \leq 1 \text{ and } x, x+t \in I, \quad (5.4)$$

where  $c_7$  is a positive constant and  $\sigma_I^2$  is the 2-density function of  $\mu$  on  $I$ .

*Proof.* By assumption we see  $\inf_{x \in I} \mathbf{m}(x) > 0$  for some open bounded, nonempty interval  $I$  in  $\mathbf{E}$ . By (4.17) we have

$$\sigma_I^2(x, x+t) \geq \rho_2(x, x+t) - \int_I \rho_3(x, x+t, z) dz \quad (5.5)$$

By (2.3) and (2.12) there exist positive constants  $c_8$  and  $c_9$  such that

$$\begin{aligned} c_8 t^\alpha &\leq \rho_2(x, x+t) && \text{for all } |t| \leq 1 \text{ and } x, x+t \in I \\ \rho_3(x, x+t, z) &\leq c_9 t^\alpha && \text{for all } |t| \leq 1 \text{ and } x, x+t, z \in I. \end{aligned} \quad (5.6)$$

Hence by taking  $I$  so small we deduce (5.4) from (5.5) and (5.6).  $\square$

*Proof of Theorem 2.4.* The closability follows from Proposition 2.9. So it only remains to prove (2.13).

Let  $(\mathcal{E}^2, \mathcal{D}^2)$  and  $(\mathcal{E}, \mathcal{D})$  denote closures of  $(\mathcal{E}^2, \mathcal{D}_\infty)$  and  $(\mathcal{E}, \mathcal{D}_\infty)$  on  $L^2(\Theta, \mu)$ , respectively. Then

$$(\mathcal{E}^2, \mathcal{D}^2) \leq (\mathcal{E}, \mathcal{D}) \quad (5.7)$$

Let  $I$  be as in Lemma 5.3. Let  $\{I_r\}_{r=1, \dots}$  be an increasing sequence of open intervals in  $\mathbf{E}$  such that  $I_1 = I$  and  $\cup_r I_r = \mathbf{E}$ . Let

$$\mathcal{E}_r^2(\mathbf{f}, \mathbf{g}) = \int_{\Theta^2} \sum_{x_i \in I_r} \frac{1}{2} \frac{\partial f(\mathbf{x})}{\partial x_i} \cdot \frac{\partial g(\mathbf{x})}{\partial x_i} d\mu \quad (5.8)$$

Here we set  $\mathbf{x} = (x_1, \dots)$ ,  $f$  and  $\mathbf{f}$  similarly as in (2.8). Then since density functions on  $I_r$  are continuous, we see  $(\mathcal{E}_r^2, \mathcal{D}_\infty)$  are closable on  $L^2(\Theta, \mu)$ . So we denote its closure by  $(\mathcal{E}_r^2, \mathcal{D}_r^2)$ . It is clear that  $\{(\mathcal{E}_r^2, \mathcal{D}_r^2)\}$  is increasing in the sense that  $\mathcal{D}_r^2 \supset \mathcal{D}_{r+1}^2$  and  $\mathcal{E}_r^2(\mathbf{f}, \mathbf{f}) \leq \mathcal{E}_{r+1}^2(\mathbf{f}, \mathbf{f})$  for all  $\mathbf{f} \in \mathcal{D}_{r+1}$ . So we denote its limit by  $(\check{\mathcal{E}}^2, \check{\mathcal{D}}^2)$ . It is known ([5, Remark (3) after Theorem 3]) that

$$(\check{\mathcal{E}}^2, \check{\mathcal{D}}^2) \leq (\mathcal{E}^2, \mathcal{D}^2). \quad (5.9)$$

By (5.7), (5.9) and the definition of  $\{(\mathcal{E}_r^2, \mathcal{D}_r^2)\}$  we conclude  $(\mathcal{E}_1^2, \mathcal{D}_1^2) \leq (\mathcal{E}, \mathcal{D})$ , which implies

$$\text{Cap}_1^2 \leq \text{Cap}, \quad (5.10)$$

where  $\text{Cap}_1^2$  and  $\text{Cap}$  denote capacities of  $(\mathcal{E}_1^2, \mathcal{D}_1^2)$  and  $(\mathcal{E}, \mathcal{D})$ , respectively. Let  $\mathbf{B} = \Theta^2 \cap \{\theta(\{x\}) = 2 \text{ for some } x \in I\}$ . Then by (2.11) and (5.4) together with a standard argument (see [2, Example 2.2.4] for example) we obtain

$$0 < \text{Cap}_1^2(\mathbf{B}). \quad (5.11)$$

Since  $\mathbf{B} \subset \mathbf{A}$ , we deduce  $0 < \text{Cap}(\mathbf{A})$  from (5.10) and (5.11), which implies (2.13).  $\square$

## 6 A construction of infinite volume dynamics

In this section we prove Theorem 2.5. We first prove the closability of pre-Dirichlet forms in finite volume.

**Lemma 6.1.** *Let  $I_r = (-r, r) \cap \mathbb{E}$  and  $\sigma_r^n$  denote the  $n$ -density function on  $I_r$ . Then  $\sigma_r^n$  is continuous.*

*Proof.* Let  $M = \sup_{x,y \in I_r} |\mathbf{K}(x,y)|$ . Then  $M < \infty$  because  $\mathbf{K}$  is continuous. Let  $\mathbf{x}_i = (\mathbf{K}(x_i, x_1), \mathbf{K}(x_i, x_2), \dots, \mathbf{K}(x_i, x_n))$  and  $\|\mathbf{x}_i\|$  denote its Euclidean norm. Then by (2.3) we see

$$|\rho_n| \leq \prod_{i=1}^n \|\mathbf{x}_i\| \leq \{\sqrt{n}M\}^n. \quad (6.1)$$

By using Stirling's formula and (6.1) we have for some positive constant  $c_{10}$  independent of  $k$  and  $M$  such that

$$\begin{aligned} \left| \frac{(-1)^k}{k!} \int_{I_r^k} \rho_{n+k}(x_1, \dots, x_{n+k}) dx_{n+1} \cdots dx_{n+k} \right| \\ \leq c_{10}^k k^{-k+1/2} (n+k)^{(n+k)/2} M^{n+k}. \end{aligned} \quad (6.2)$$

This implies for each  $n$  the series in the right hand side of (4.17) converges uniformly in  $(x_1, \dots, x_n)$ . So  $\sigma_r^n$  is the limit of continuous functions in the uniform norm, which completes the proof.  $\square$

**Lemma 6.2.**  *$(\mathcal{E}, \mathcal{D}_{\infty, r})$  are closable on  $L^2(\Theta, \mu)$ .*

*Proof.* Let  $I_r = \{x \in \mathbb{E}; |x| < r\}$  and  $\Theta_r^n = \{\theta(I_r) = n\}$ . Let  $\mathcal{E}_r^n(\mathbf{f}, \mathbf{g}) = \int_{\Theta_r^n} \mathbb{D}[\mathbf{f}, \mathbf{g}] d\mu$ . Then it is enough to show that  $(\mathcal{E}_r^n, \mathcal{D}_{\infty, r})$  are closable on  $L^2(\Theta, \mu)$  for all  $n$ .

Since  $\mathbf{f}$  is  $\sigma[\pi_r]$ -measurable, we have  $(\mathbf{x} = (x_1, \dots, x_n))$

$$\mathcal{E}_r^n(\mathbf{f}, \mathbf{g}) = \frac{1}{n!} \int_{I_r^n} \sum_{i=1}^n \frac{1}{2} \frac{\partial f_r^n(\mathbf{x})}{\partial x_i} \cdot \frac{\partial g_r^n(\mathbf{x})}{\partial x_i} \sigma_r^n(\mathbf{x}) d\mathbf{x},$$

where  $f_r^n$  and  $g_r^n$  are defined similarly as after (4.15). Then since  $\sigma_r^n$  is continuous, we see  $(\mathcal{E}_r^n, \mathcal{D}_{\infty, r})$  is closable.  $\square$

*Proof of Theorem 2.5.* By Lemma 6.2 we see the assumption (A.1\*) in [5] is satisfied. (A.2) in [5] is also satisfied by the construction of determinantal random point fields. So one can apply results in [5] (Theorem 1, Corollary 1, Lemma 2.1 (3) in [5]) to the present situation. Although in Theorem 1 in [5] we treat  $(\mathcal{E}, \mathcal{D})$ , it is not difficult to see that the same conclusion also holds for  $(\mathcal{E}^{\text{reg}}, \mathcal{D}^{\text{reg}})$ , which completes the proof.  $\square$

## 7 Gibbsian case

In this section we consider the case  $\mu$  is a canonical Gibbs measure with interaction potential  $\Phi$ , whose  $n$ -density functions for bounded sets are bounded, and 1-correlation function is locally integrable. If  $\Phi$  is super stable and regular in the sense of Ruelle, then probability measures satisfying these exist. In addition, it is known in [5] that, if  $\Phi$  is upper semi-continuous (or more generally  $\Phi$  is a measurable function dominated from above by a upper semi-continuous potential satisfying certain integrable conditions (see [7])), then the form  $(\mathcal{E}, \mathcal{D})$  on  $L^2(\Theta, \mu)$  is closable. We remark these assumptions are quite mild. In [5] and [7] only *grand* canonical Gibbs measures with *pair* interaction potential are treated; it is easy to generalize the results in [5] and [7] to the present situation.

**Proposition 7.1.** *Let  $\mu$  be as above. Assume  $d \geq 2$ . Then  $\text{Cap}(\mathbf{A}) = 0$  and no collision (2.10) occurs.*

*Proof.* The proof is quite similar to the one of Theorem 2.1. Let  $\mathbf{I}_\epsilon$  be as in (4.22). It only remains to show  $\lim_{\epsilon \rightarrow 0} \mathbf{I}_\epsilon = 0$ .

We divide the case into two parts: (1)  $d = 2$  and (2)  $3 \leq d$ . Assume (1). We can prove  $\lim \mathbf{I}_\epsilon = 0$  similarly as before. In the case of (2) the proof is more simple. Indeed, we change definitions of  $A_\epsilon^+$  in (4.7) and  $h_\epsilon$  in (4.8) as follows:  $A_\epsilon^+ = A_{4\epsilon}^{\mathbf{a}}(r, n, m)$

$$h_\epsilon(t) = \begin{cases} 2 & (|t| \leq \epsilon) \\ -(2/\epsilon)|t| + 4 & (\epsilon \leq |t| \leq 2\epsilon) \\ 0 & (2\epsilon \leq |t|). \end{cases} \quad (7.1)$$

Then we can easily see  $\lim \mathbf{I}_\epsilon = 0$ . □

*Remark 7.2.* (1) This result was announced and used in [6, Lemma 1.4]. Since this result was so different from other parts of the paper [6], we did not give a detail of the proof there.

(2) In [8] a related result was obtained. In their frame work the choice of the domain of Dirichlet forms may be not same as ours. Indeed, their domains are smaller than or equal to ours (we do not know they are same or not). So one may deduce Proposition 7.1 from their result.

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